Introduction to Fourier Series

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If you are curious about the Fourier series and how it was used in the analytical description of heat diffusion in solids, Here are some notes from "The Analytical Theory of Heat" by Jean Baptiste Joseph Fourier. What better way to learn the concept than by learning it from the master?

To begin with, a short description of the Fourier series and what it is supposed to do. The series which goes by his name nowadays appeared in Chapter 3 of Fourier's classic The Analytical Theory of Heat, which appeared in print in 1822 for the first time.

If there is a function $f(x)$ then Fourier tells us that it can be defined within the range $[-\infty, +\infty]$ using an infinite summation of the trigonometric functions $sin(x)$ and $cos(x)$.

In an equation form it would be represented as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx) \tag{1}$$

We know $f(x)$ represents a curve in the $x - f(x)$ space. Suppose, if $f(x) = mx$, we know the curve is a straight line. And we know that this mx can be easily represented using the RHS of Eq. (1) if we can suitably 'tweak' the constants. Similarly, if $f(x)$ is $sin(x)$ or $cos(x)$ or e^x or simply 1, we know how to tweak the constants on the RHS of Eq. (1) to make it work. But what Fourier did was, for an arbitrary function $f(x)$, he proved that the series representation in Eq. (1) holds true and also showed a way to find the constants as

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[1] intended as course notes.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) dx \tag{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) cos(nx) dx \tag{3}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) sin(nx) dx \tag{4}$$

That is, to find the coefficient a_0, Fourier integrates Eq. (1) on both sides within the range $[-\infty, +\infty]$ to obtain Eq. (2); to obtain a_n, he multiplies Eq. (1) first on both sides with $cos(mx)$ and then integrates over the range $[-p, +p]$ to obtain Eq. (3) and follows the same procedure for b_n by using the $sin(mx)$ as the multiplication factor to obtain Eq. (4).

Two assumptions that Fourier made in the above scheme are the integrability of $f(x)$ (else one wouldn't know the answers for the coefficients) and the integral of the infinite sums was identical to the infinite sum of the integrals as follows

$$\int_{-\pi}^{+\pi} \left[\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx) \right] dx$$
$$\equiv \frac{1}{2} a_0 + \int_{-\pi}^{+\pi} \left[a_n cos(nx) + b_n sin(nx) \right] dx + \frac{1}{2} a_0 \int_{-\pi}^{+\pi} dx \tag{5}$$

Daniel Bernoulli a century earlier in 1740s has proposed such an infinite series expression as in Eq. (1) while developing a theory for vibrating musical strings but was not able to find the coefficients (Eqs. 2 to 4). According to Stephen Hawking [1], the determination of the coefficients (Eqs. 2 to 4) was the greatest accomplishment of Fourier. According to Lord Kelvin,

> Fourier's theorem is not only one of the most beautiful results of modern analysis, but it is said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics...Fourier is a mathematical poem.

[The above quote was taken from the recent compilation by Stephen Hawking of the works of thirty greatest mathematicians [1] according to him]

Just to give an example, if $f(x)$ is such that it represents the periodic triangle like curve depicted in white color in Fig. 1 and the red color curve, the result of using Eq. (1) to approximate the actual function $f(x)$.

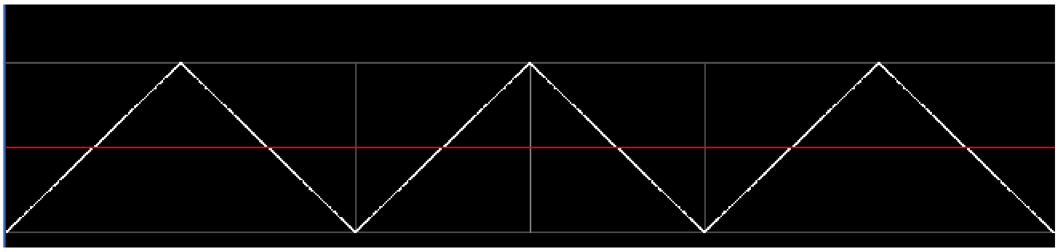

Fig. 1.

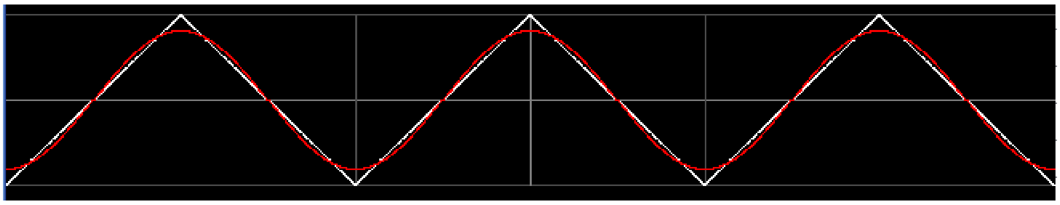

Fig. 2.

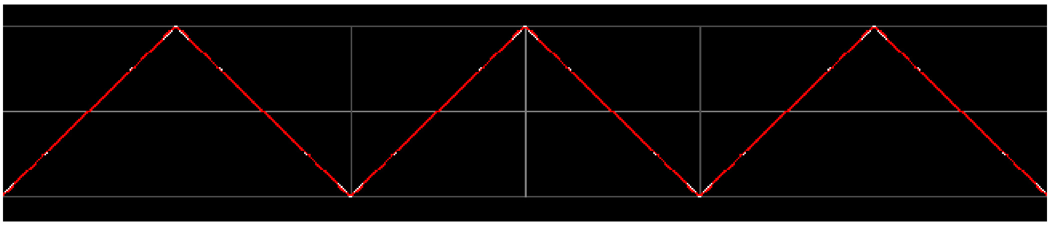

Fig. 3.

Using a pair of sine and cosine (in the infinite summation of RHS of Eq. (1)) would get us to Fig. 2 already

And with about 13 to 14 sets of sine and cosines, we can obtain with fair exactitude, the required triangle like curve as in Fig. 3.

In other words, a few suitable summation of wiggly curves lead us to a rigid choppy curve - all courtesy Fourier, who showed it can be done by 1822.

You can play with this Java applet `http://www.falstad.com/fourier/`.

Now, how did Fourier use Eq. (1) to explain heat diffusion?

Reference

[1] God Created the Integers, edited with commentary by Stephen Hawking, Penguin, 2005.