

First Law and Fourier Law

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First Law of thermodynamics relates heat transfer, work transfer and internal energy of a body. Fourier Law of heat conduction proposes how heat transfers in a solid body. We shall heat a block and see how hot the understanding gets.

Observe the figure below. But for the faces at $x = 0$ and $x = L$, the rest of the faces (four of them) are "insulated" such that energy cannot cross into or out of this block - our system of interest - as heat. Add to this we are not doing energy transfer as work on this block (like pushing or pulling it; squeezing or shearing it; electrocuting or "magneto-cutting" it; you get the idea). Also, the block isn't doing any work interaction on us - the surrounding.

Applying the First Law of Thermodynamics on this block, since it is a closed system that allows energy interaction with the surrounding (we are heating it, for instance) but not mass flow in or out (the block remains as a solid while we heat and investigate), we can put succinctly what happens to the block as

$$\delta Q - \delta W = dE \quad (1)$$

The above equation says the difference in the sum of heat transfer interactions and the sum of work transfer interactions for the block should be equal to the change in the total energy of the block. Or, if there are no energy interactions from or into the block, then the sum total of the energy of the block remains constant. All the terms in Eq. (1) are hence expressed in Joules, the unit of measure for Energy in SI units.

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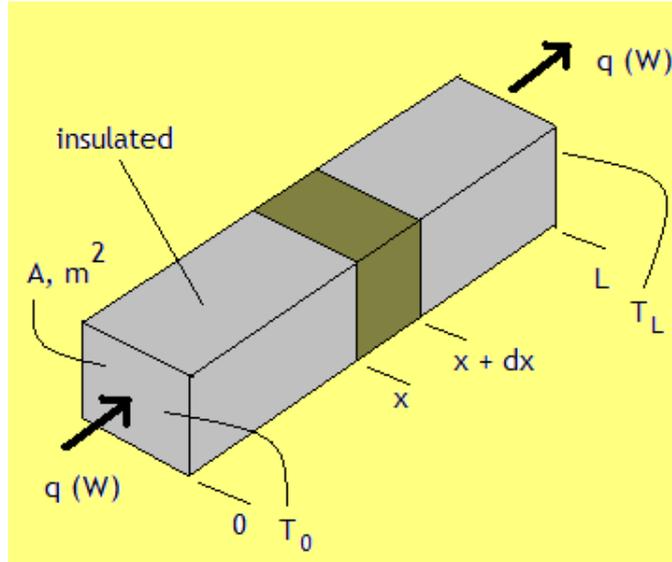


Fig. 1.

Now we have an equation that characteristically doesn't contain information about the time involved for the energy interaction processes [as explained here, in an earlier essay]. To bring in this information, let us express Eq. (1) as a rate equation, (i.e. variables expressed as divided per time) resulting in

$$q - w = \frac{dE}{dt} \quad (2)$$

In Eq. (2) above each of the terms is expressed in Watts (= Joules / second). Let us take the small elemental slab from the block (the darkened volume in Fig. 1 lying between x and $x + dx$) and assume two heat transfer interactions take place across the surface area A of the block, one at location x and the other one at $(x + dx)$. As we assume until now no work interactions exist for the block, we can nullify the second term on the LHS of Eq. (2). This would result in

$$q_x - q_{x+dx} = \frac{dE}{dt} \quad (3)$$

At this stage, if one were to know how to measure the RHS, i.e. the time rate of change of the total energy of the block, then one has a way to measure of what is the heat transfer interaction happening to the block (heated or cooled or neither). The vice versa situation where if one were to measure the LHS, i.e. the heat transfer interactions in time, it would allow one to know of the energy inventory of the block in time.

Unfortunately, neither of this could be done. We neither have an energy

meter, so to say, that directly measures the RHS nor a heat meter, so to say, that directly measures the LHS. So we model and simplify and relate the non-measurable to the measurable. Since the block is a thermodynamic closed which precludes bulk mass exchange with its surroundings (minor few molecules escaping out here and there from the block is excluded in this continuum approach), one can express the RHS of Eq. (3) in a derived form. For doing this we should know we can express the total energy E as

$$E = \rho(Adx)u \quad (4)$$

where u is the specific (per unit mass) internal energy of the block and the product of the rest of the symbols in RHS should result in a mass of the block (density times and elemental volume - see Fig. 1). This is possible because we can safely assume the block in Fig. 1 to have negligible Kinetic and Potential energies, at least during the heating or cooling process is going on, just to focus our attention only on that. This results in the total energy to depend only on the change in the internal energy of the block. Further, the change in this internal energy, being something that is not accounted for in KE and PE, can be related to the change in the temperature of the block as

$$du = c \cdot dT \quad (5)$$

where c is the specific heat of the block material. Using the above two simplifications, the RHS of Eq. (3) can be rewritten as

$$\frac{\partial E}{\partial t} = \rho c(Adx) \frac{\partial T}{\partial t} \quad (6)$$

Now for identifying the measurable quantities. Equation (6) involves in the RHS the density, specific heat, the cross section area of the block (see Fig. 1), the length along which the heat current travels and the change in the temperature in time. All of these are measurable with fair accuracy using our existing technology, at least for the block in consideration. Since the block is a solid, an incompressible substance, one doesn't have to wonder about whether this specific heat is measured at constant pressure or constant volume. Both of those experiments would result in identical values for c for solids (for gases these two specific heats will be different). Temperature can be measured using a thermometer or a thermocouple.

Now if one were to use Eq. (6) to measure indirectly (i.e. by measuring the RHS of Eq. (6)) the change in the total energy of the block (the LHS of Eq. (6)) and substitute this in Eq. (3), one could obtain the heat interaction in time for the block. Since Eq. (6) is still in its differential form, we shall keep

it that way for the moment and proceed to look at what we can do with the LHS of Eq. (3), the heat interaction.

We are struck with measuring q directly. Unless we know how to do this, there ends, perhaps prematurely, what First Law can tell us about the heating of the block.

Enter Jean Baptiste Joseph Fourier (some argue it should be Enter Jean Baptiste Biot). He showed a way out with his creativity, by assuming the local heat current q to be proportional to the local spatial temperature difference, as in for location x ,

$$q_x \propto -\frac{\partial T}{\partial x} \quad (7)$$

This proportionality in (7) was later resolved using experiments so that Eq. (7) is written nowadays as

$$q_x = -k \cdot A \cdot \frac{\partial T}{\partial x} \quad (8)$$

where 'k' is a material property (of the block, in our case) called the thermal conductivity. Equation (8) is called the constitutive equation for k because that is the one which defines what k is and hence using only which one could measure k (as done in experiments to measure k).

However, what Eq. (8) essentially does for us is that it removes the impediment of having to measure q directly and, as suggested by Fourier, allows us to find q by measuring only the local temperatures - using thermometers or thermocouples.

By the way, the negative sign in front of the RHS term indicates the heat current flows in the direction of the negative gradient of the temperature, i.e. from higher temperature to lower temperature. In the Fig. 1, temperature is assumed to decrease along the positive x , the direction of the heat current vector.

Equation (8), the Fourier Law, is distinct from Eq. (2) or (3), the First Law of thermodynamics written for the block. Proceeding further in rewriting the LHS of Eq. (3), we can see the q at location $(x + dx)$ can be written using a Taylor series expansion truncated to the first two terms as

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx \quad (9)$$

What we say in Eq. (9) is that we can find the q at location $(x + dx)$, very small dx distance away from x as the q at location x and its spatial change between x and $(x + dx)$ times the distance dx itself. Simple, isn't it? Of course, the second order variation as in the change in the change of the q at x can be found and added to this, but we assume these variations to be minimal when compared to the first derivative change and neglect them. This is what we meant by saying we truncate the Taylor's series with the first two terms itself.

Now applying Fourier's assumption at location $(x + dx)$, we could easily rewrite Eq. (9) as

$$q_{x+dx} = -k \cdot A \cdot \frac{\partial T}{\partial x} + \frac{\partial}{\partial x} \left(-k \cdot A \cdot \frac{\partial T}{\partial x} \right) dx \quad (10)$$

Combining Eq. (6), Eq. (8), Eq. (10) and Eq. (3) and rearranging and canceling out some terms appearing on both LHS and RHS, we can write the First Law as applied for the block in Fig. 1, in terms of measurable quantities as

$$\frac{\partial}{\partial x} \left(k \cdot \frac{\partial T}{\partial x} \right) = \rho c \frac{\partial T}{\partial t} \quad (11)$$

To make this equation even more general, now we can add one more missing variable, the work term that we kept until now as zero. Work interaction can happen in a block such as that in Fig. 1 by means of stretching or shrinking of the block or by passing electricity through it. Let us say we pass electricity through the block. By doing this we are doing work on the block system. The result of this work transfer interaction is to add energy to the block. However, the block could either store it or release it back to the surrounding. This means, it has to be accounted for in the First Law energy balance, as written in Eq. (2). The work interaction of passing an electric current through the block is modeled usually as a heat generation quantity measured in Watts per unit volume of the block. This heat generation is understood to have its origin in the ohmic losses that the electric current exhibits, resulting in that losses to dissipate out of the block as heat transferred to the surroundings. In other words, the block should get hotter because of electric current passage, if it doesn't throw out this heat "generated".

Another example where direct heat generating is involved is that of a fissile nuclear material, the fuel inside a nuclear reactor, which generates heat internally because of nuclear fission - which, in turn, we take out by means of an heat exchanger device to generate steam and run turbines to generate electricity.

We are now in a position to write First Law of Thermodynamics for the closed

system of the block shown in Fig. 1 as

$$\frac{\partial}{\partial x} \left(k \cdot \frac{\partial T}{\partial x} \right) + \dot{q} = \rho c \frac{\partial T}{\partial t} \quad (12)$$

The first term in the LHS of the above Eq. (12) is the longitudinal heat conduction that is the resultant difference between the q at x and $(x + dx)$ location in Fig. 1. The second term is the heat generation, in principle, the work interaction term. Both of these terms bring in or take out energy from the block. The total of these two should be balanced by the energy inventory of the block, represented by the RHS term, also known as the thermal inertia of the block. That is, this term determines how hot the block should get to, for a given energy interaction (on the LHS). Lesser the thermal capacity (density times specific heat), quicker the temperature raise of the block.

Obviously this means, in steady state, where we mean the temperature anywhere inside the block is not changing with time, the RHS should go to zero. Also, if there is no heat generation involved for the block, the second term of Eq. (12) also goes to zero. In other words, the steady state First Law energy balance for the block in the elemental volume between x and $(x + dx)$ simplifies to

$$\frac{d^2T}{dx^2} = 0 \quad (13)$$

This means, in Fig. 1, the energy that enters as heat current q at x (in Watts) promptly leaves at the location $(x + dx)$, leaving no trace on the energy inventory of the elemental block volume, so that it doesn't register a temperature increase in time.

If we write this for all the three directions x , y and z in Cartesian coordinates, then it would look like,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \quad (14)$$

and if you believe in brevity as the wit of the mathematics soul, then we can express Eq. (14) also as

$$\nabla^2 T = 0 \quad (15)$$

Remember, we have only formulated the First Law statement in differential form and are yet to solve this equation. Using the conditions of all the four faces being insulated but for the left and right for the block in Fig. 1, we can safely assume the heat transfer interaction in the block to be one dimensional so that we can use Eq. (13) for analyzing its steady state behavior. To solve Eq. (13), we require at least two conditions, i.e. two correct answers about how temperature behaves with respect to space coordinate x . These two conditions are given for us in Fig. 1, if we take the temperature of the block at $x = 0$, and $x = L$ to be known. In other words, we are measuring the temperatures at these two locations. This allows us to solve Eq. (13), the solution of which looks like

$$T = T_0 + (T_L - T_0) \cdot (x/L) \quad (16)$$

Using this information, we can find out how exactly the Fourier assumption in Eq. (8) looks like. Finding the first derivative of T from Eq. (15) and using it in Eq. (8) we obtain

$$q'' = \frac{k}{L} (T_0 - T_L) \quad (17)$$

Now the heat flux q'' (Watts per square meter - q across area A in Fig. 1) can be independently measured by other means. Since the temperatures on the RHS of Eq. (17) are measured at locations $x = 0$ and $x = L$, the only unknown k the thermal conductivity of the material of the block can be determined this way.

And for doing it, we now have cleverly coupled in Eq. (17), the distinct results of First Law, Eq. (3) and Fourier Law, Eq. (8).

Major References

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