Introduction to Scale Analysis

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Scale analysis or order of magnitude analysis is a back of the envelope [1] method for deriving the most information with a reasonable understanding of the physics of a phenomenon. In other words, it is a method to yield first cut estimates about the relationships between several parameters involved in a particular problem. Let us take an example [2] to see how this works.

Suppose we have a thin plate of transverse width D (see figure) is immersed in a hotter fluid at time t = 0. The question we have is at what time the center of the plate will "feel the heat" so that the temperature of that location starts rising from then on, from its initial temperature. Quenching [4], a process of engineering interest, is the reverse of this process, where the plate is "cooled" by the fluid.

To answer this question let us simplify things a bit and assume that the height and thickness of the plate are such that one can consider an unsteady (changing in time) conduction heat transfer process to govern the temperature increase in the plate. The one dimensional heat equation governing the transient conduction heat transfer process in this thin plate is then given by

$$c_P \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{1}$$

where T is the temperature, t the time, x the spatial position, c_P is the specific heat capacity at constant pressure, k the thermal conductivity and symbol ρ, the density of the material that undergoes the transient conduction heat transfer.

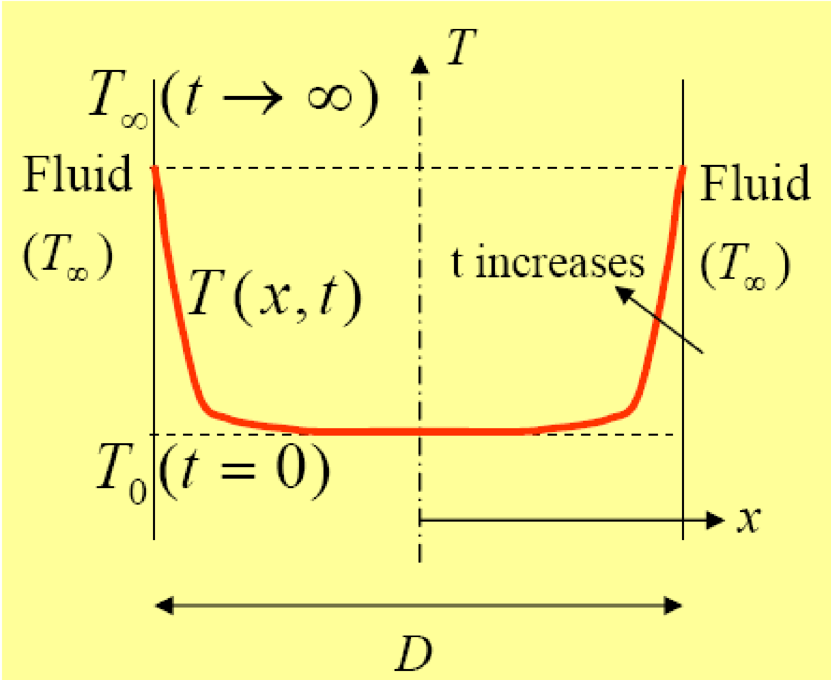

Fig. 1. Schematic of heating a thin plate by hot fluid

Owing to the symmetry of the situation (see figure) we can concentrate on one half of the plate (D/2) for making an order of magnitude estimate of each term in Eq. (1) above. The LHS of Eq. (1) for instance can be *scaled* as

$$c_P \frac{\partial T}{\partial t} \sim \rho c_P \frac{\Delta T}{t} \tag{2}$$

Observe the *tilda* instead of the equality symbol in Eq. (2). Also, the delta T in Eq. (2) is the temperature difference possible for the system (Fig. 1) in the time t (unknown) and is unknown as such and we can make a guess about it once we write the scale for the RHS of Eq. (1) as well. Proceeding to do so, the RHS of Eq. (1) can be *scaled* as

$$k \frac{\partial^2 T}{\partial x^2} = k \frac{\partial}{\partial x} \left(\frac{\partial T}{\partial x} \right) \sim k \frac{1}{(D/2)} \frac{\Delta T}{(D/2)} = k \frac{\Delta T}{(D/2)^2} \tag{3}$$

Observe in Eq. (3) as well there appears a ΔT whose exact value is unknown. Using Eqs. (2) and (3) in (1), we can find a time scale as

$$t \sim \frac{(D/2)^2}{\alpha} \tag{4}$$

where α is the thermal diffusivity of the material of the plate.

If Eq. (4) is the answer to our problem of when the center of the plate will start feeling the heat or will begin to rise its temperature, then the two unknown ΔT values in Eq. (2) and (3) must be equal. In fact, it is the only instance when the ΔT across the spatial distance of $D/2$ (see figure) will match exactly with the ΔT for the time duration t (measured from the initial time of $t = 0$). Hence our assumption about ΔT while finding the answer in Eq. (4) is correct. The result in Eq. (4) for determining the heat penetration time compares well with the results from the exact analysis to the problem.

And the solution method is that simple.

Let us proceed with some more observations on the basic rules [2] for using this technique on other situations/equations.

It is necessary to define limits for the spatial region in which we perform the analysis. For instance, in the above example the x-axis is extended between 0 and $D/2$ (and not beyond that).

The philosophy behind this technique is every equation, in a physical sense, is a balance between two dominant scales resulting in the effect or phenomenon that equation is trying to describe. If this is not so, that is, if there are more than a few terms in an equation, one must find out the dominant terms for a particular situation so that the equation could be interpreted (reduced) as a balance between two terms.

For identifying the dominant scales, in general, these rules could be used

For an equation of the form $c = a + b$, if the order of magnitude of one term is greater than the other, i.e. $O(a) > O(b)$, then the order of magnitude of the sum c = a + b is determined by the dominant term, i. e. $c = a + b$ can be interpreted in a scaling sense as $O(c) = O(a)$. Similar conclusions hold for $c = a - b$.

If on the other hand, in an equation c = a + b, O(a) = O(b), then one term cannot be thrown out in comparison with the other. Both of them have to be treated with respect and the only scaling argument one can derive is $O(c) \sim O(a) \sim O(b)$.

In a product c = ab, the small is as weighty as the big and so the only scaling rule is of the form O(c) = O(a)O(b). Similar arguments hold for fractions (c = a/b) as well.

While interpreting the result from a scale analysis, one should be careful about certain things. For instance, the result of such an analysis will be accurate only to an order of magnitude, i.e. if the result predicts a scale of order 1, the actual answer could be anywhere within the next order on both sides (0.1 to 10). This may sometimes lead to erroneous and confusing interpretations. For instance if the result from a scale analysis suggests certain parameter group is of order one, say, $C \sim 1$ and if the actual answer is found to be $C^2 = 0.1$, it doesn't mean the scaling analysis is necessarily wrong. The correct interpretation lies in the fact that since the scale analysis yields a result of order one for C, it allows C to be even less than 1, which when squared (i.e. C^2) could lead to an answer (the correct one that matches with the exact answer, in this case) that is even smaller.

For instance, such an occasion appears when one compares the results from scaling arguments with the more exact results for the prediction of the thermal entrance length in a duct flow exhibiting hydrodynamic and thermal boundary layer growth. The scaling could still be correct; only the interpretation isn't.

On the other hand, scale analysis as a solution technique finds more use and relevance with differential equations rather with integral equations, a reason for its lack of use in such fields as radiation heat transfer, wherein the phenomenon is modeled more with integral and/or integro-differential equations.

Scale analysis or order of magnitude analysis is practiced both as a research and pedagogical tool for more than a century. The earliest example in the field of fluid flow could be the order of magnitude analysis performed by Ludwig Prandtl [5] to reduce the formidable Navier-Stokes equations [6] into a slightly less formidable boundary layer equations, valid inside a boundary layer [6] - the region of fluid flow near a solid wall, where viscous effects dominate in the fluid.

Many researchers have used this scale analysis with remarkable success in delineating good physical insights about complex problems of scientific relevance. Some applications of this method to a wide spectrum of problems are discussed in [3].

References

(1) `http://en.wikipedia.org/wiki/Back_of_the_envelope`
(2) the example and the basic rules are discussed in Convection Heat Transfer, by A. Bejan [Amazon Link] (the same example is also discussed in [3] below. Some observations are mine).
(3) Qualitative Methods in Physical Kinetics and Hydrodynamics, by Vladimir Krainov [Amazon LInk].
(4) `http://en.wikipedia.org/wiki/Quenching`
(5) `http://en.wikipedia.org/wiki/Ludwig_Prandtl`
(6) `http://en.wikipedia.org/wiki/Navier-Stokes_equation`
(7) `http://en.wikipedia.org/wiki/Boundary_layer`