Scale Analysis of Flat Plate Forced Convection

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Forced convection flow [1] over a flat plate with a leading edge is a basic convection configuration that has yielded to analytical scrutiny. A coupled system of partial differential equations - Navier Stokes equations [2] plus First Law of Thermodynamics [3] applied on a control volume - are solved to find the chief unknown, the convection heat transfer coefficient.

Scale analysis has been discussed earlier [4]. The following explains scale analysis performed on the boundary layer simplified equations, in the thick and thin thermal boundary layer limits of forced convection flow over an isothermal hot flat plate. Text books [5] contain the discussion in varied level of detail. This note fills minor missing details.

By habit, the flow is non-Arabic - is depicted from left to right, as shown in the accompanying schematic.

The free stream flow entering the control volume of the flat plate at the leading edge on the left (not shown) at an uniform (spatially identical) velocity U_∞ is deccelerated to zero velocity by the stationary hot flate plate. The plate is at an isothermal higher temperature T_W than the temperautre T_∞ of the incoming and free stream flow. For the purpose of this discussion, it is assumed that the solution to the viscous fluid flow equations describing the steady, laminar, incompressible flow is given. Hence the hydrodynamic boundary layer (HBL) thickness (δ) is known. If necessary, read about the concept of hydrodynamic boundary layer and thermal boundary layer in the Wikipedia [6].

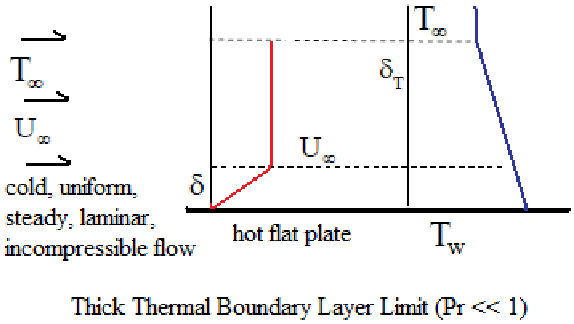

Fig. 1. Schematic for forced convection over flat plate in the thick thermal boundary layer limit

The schematic is for the thick thermal boundary layer (TBL) limit (i.e. TBL thicker than HBL). By requirement, the fluid's Prandtl number satisfies $Pr \ll 1$. The assumption of slenderness of both the boundary layers (the tenet of boundary layer concept) imply that $\delta \ll L$ and $\delta_T \ll L$ even when one of them is bigger than the other.

We begin with the thermal boundary layer (TBL) simplified energy equation in two dimension written in Cartesian coordinates:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{1}$$

That the axial diffusion (in x-dir) is negligible is the TBL simplification of the First Law of Thermodynamics. For the thick TBL shown in the figure, the temperature gradient prevails over a larger distance than that of the velocity gradient. Applying scale analysis we can write the appropriate scales for the differential equation above as

$$u\frac{\Delta T}{L} + v\frac{\Delta T}{\delta_T} = \alpha\frac{\Delta T}{\delta_T^2} \tag{2}$$

where $\Delta T = T_W - T_\infty$ and L is some finite length of the plate in x direction (it can even be just x, as in from $x = 0$ to $x = x$). Without bounding the domain this way, scale analysis won't make sense. Obviously, we need the scales for u and v the local velocities in x and y directions to proceed. These can be obtained by invoking the mass conservation, which for the situation discussed would be:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

From the schematic for thick TBL above, since the HBL is much smaller than the TBL ($\delta < \delta_T$), the scale for u can safely be written as $u \sim U_\infty$. Therefore, from the mass conservation scales for Eq. (3) within the TBL one can obtain the scale for v as

$$U_\infty L \sim \frac{v}{\delta} \Rightarrow v \sim U_\infty \frac{\delta}{L} \tag{4}$$

In other words, for much of the TBL, free stream flow prevails and the role of HBL in local convection is restricted very close to the hot flat plate. Observe also the use of δ in this scale, as the HBL ends well within the TBL.

Using the u and v scale from the above equation back in the scaled energy equation (2), one can see the LHS terms scale as

$$U_\infty \frac{\Delta T}{L}, U_\infty \frac{\delta}{L} \frac{\Delta T}{\delta_T} \tag{5}$$

In other words, the second LHS term is δ/δ_T times *smaller* than the first term and hence can be considered negligible. This results in the simplification of the TBL energy equation further as a balance between longitudinal convection by the fluid and transverse conduction heat transfer from the flat plate, written as

$$U_\infty \frac{\Delta T}{L} \sim \alpha \frac{\Delta T}{\delta_T^2} \tag{6}$$

resulting - after some substitution and rearrangement - in

$$\delta_T L \sim Pe^{-1/2} = Pr^{-1/2} Re^{-1/2} \tag{7}$$

The above equation is the physics for thick TBL when $\delta_T \gg \delta$ and $Pr \ll 1$, where $Re = U_\infty L/\nu$ and $Pr = \nu/\alpha = \delta/\delta_T$. ν is the kinematic viscosity or the momentum diffusivity and α is the thermal diffusivity of the fluid.

In the thin TBL limit, the converse holds, i.e. $\delta_T \ll \delta$ and $Pr \gg 1$ as shown in the accompanying schematic. Here, since the HBL extends well beyond the TBL, the scale for u within the TBL is not U_∞ but $u \sim U_\infty \frac{\delta_T}{\delta}$ as can be verified by the geometry of the figure. When used in the mass conservation, this modifies the scale for v as

$$v \sim U_\infty \frac{\delta_T}{L} \frac{\delta_T}{\delta} \tag{8}$$

 Source: http://www.nonoscience.info/

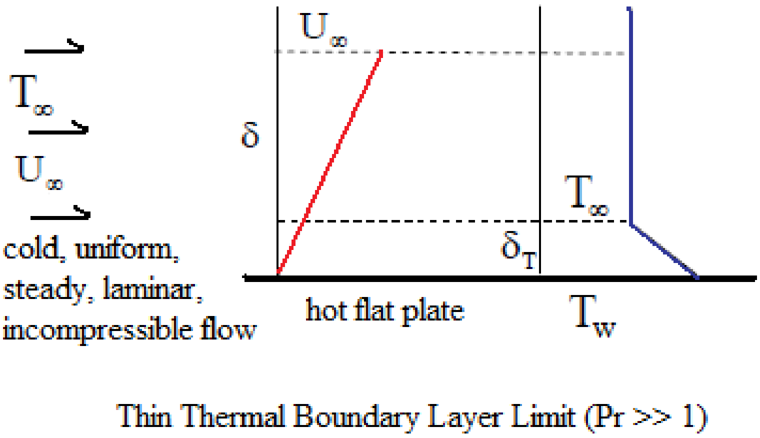

Fig. 2. Schematic for forced convection over flat plate in the thin thermal boundary layer limit

Observe the use of δ_T as a transverse scale in the above mass balance as our domain of interest is only the TBL, beyond which even though the HBL extends, there is no heat transfer (compare with mass balance for thick TBL above).

The above observation suggests that one cannot readily start with the simplified Eq. (5) as energy balance, but should verify whether the scales for the two LHS terms in Eq. (2) are comparable. Using the modified scales for u and v for the thin TBL into Eq. (2), one can rewrite it as

$$U_\infty \frac{\delta_T}{\delta} \frac{\Delta T}{L}, U_\infty \frac{\delta_T}{\delta} \frac{\delta_T}{L} \frac{\Delta T}{\delta_T} \sim \alpha \frac{\Delta T}{\delta_T^2} \tag{9}$$

The two LHS terms in the above equation is of the same order of scale. Setting one of it to balance the RHS in the convection c̃onduction balance, one obtains

$$U_\infty \frac{\delta_T}{\delta} \frac{\Delta T}{L} \sim \alpha \frac{\Delta^2 T}{\delta_T^2} \tag{10}$$

Reducing the above equation after substituting for the HBL thickness as $\delta = (\nu L / U_\infty)^{1/2}$, one can find the TBL thickness as

$$\delta_T L \sim Re^{-1/2} \cdot Pr^{-1/3} \tag{11}$$

a result that shows the distinct change in the Pr effect, when compared to Eq. (7) for thick TBL limit.

For completion, using the TBL thickness of the TBL one can find the local (at a x-location, in the figures) convection heat transfer coefficient, expressed in non dimensional form as the Nusselt number of the configuration (in honour of Nusselt) as

$$Nu_x \sim Pr^{1/2} \cdot Re^{1/2} \tag{12}$$

for $\delta \ll \delta_T$ and $Pr \ll 1$ and

$$Nu_x \sim Pr^{1/3} \cdot Re^{1/2} \tag{13}$$

for $\delta \gg \delta_T$ and $Pr \gg 1$.

From the above two equations it is evident that when $Pr \sim 1$, the heat transfer depends only on the square-root of the local Re.

References

(1) Free and Paid Convection http://www.nonoscience.info/2006/07/10/free-and-paid-convection/
(2) Navier Stokes Equation http://en.wikipedia.org/wiki/Navier-Stokes_equation
(3) First Law and Fourier Law http://www.nonoscience.info/2008/07/12/first-law-and-fourier-law/
(4) Scale Analysis http://www.nonoscience.info/2008/07/14/scale-analysis/
(5) the basic analysis is discussed in Convection Heat Transfer, by A. Bejan [Amazon Link]
(6) Boundary Layer http://en.wikipedia.org/wiki/Boundary_layer